Likelihood calculations for *vsn*

Wolfgang Huber

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Contents

1 Introduction

This vignette contains the computations that underlie the numerical code of *vsn*. If you are a new user and looking for an introduction on how to **use** *vsn*, please refer to the vignette *Robust calibration and variance stabilization with vsn*, which is provided separately.

2 Setup and Notation

Consider the model

$$\operatorname{arsinh}\left(f(b_i) \cdot y_{ki} + a_i\right) = \mu_k + \varepsilon_{ki} \tag{1}$$

where μ_k, for $k = 1, \ldots, n$, and a_i, b_i, for $i = 1, \ldots, d$ are real-valued parameters, f is a function $\mathbb{R} \to \mathbb{R}$ (see below), and ε_{ki} are i.i.d. Normal with mean 0 and variance σ^2. y_{ki} are the data. In applications to μarray data, k indexes the features and i the arrays and/or colour channels.

Examples for f are $f(b) = b$ and $f(b) = e^b$. The former is the most obvious choice; in that case we will usually need to require $b_i > 0$. The choice $f(b) = e^b$ assures that the factor in front of y_{ki} is positive for all $b \in \mathbb{R}$, and as it turns out, simplifies some of the computations.

In the following calculations, I will also use the notation

$$Y \equiv Y(y, a, b) = f(b) \cdot y + a \tag{2}$$

$$h \equiv h(y, a, b) = \operatorname{arsinh}\left(f(b) \cdot y + a\right). \tag{3}$$

The probability of the data $(y_{ki})_{k=1\dots n,\, i=1\dots d}$ lying in a certain volume element of y-space (hyperrectangle with sides $[y_{ki}^{\alpha}, y_{ki}^{\beta}]$) is

$$P = \prod_{k=1}^{n}\prod_{i=1}^{d}\int_{y_{ki}^{\alpha}}^{y_{ki}^{\beta}} dy_{ki}\;\; p_{\text{Normal}}(h(y_{ki}), \mu_k, \sigma^2)\;\frac{dh}{dy}(y_{ki}), \tag{4}$$

where μ_k is the expectation value for feature k and σ^2 the variance.

With

$$p_{\text{Normal}}(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{5}$$

the likelihood is

$$L = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^{nd} \prod_{k=1}^{n}\prod_{i=1}^{d} \exp\left(-\frac{(h(y_{ki})-\mu_k)^2}{2\sigma^2}\right)\cdot\frac{dh}{dy}(y_{ki})\,. \tag{6}$$

For the following, I will need the derivatives

$$\frac{\partial Y}{\partial a} = 1 \tag{7}$$

$$\frac{\partial Y}{\partial b} = y\cdot f'(b) \tag{8}$$

$$\frac{dh}{dy} = \frac{f(b)}{\sqrt{1+(f(b)y+a)^2}} = \frac{f(b)}{\sqrt{1+Y^2}}, \tag{9}$$

$$\frac{\partial h}{\partial a} = \frac{1}{\sqrt{1+Y^2}}, \tag{10}$$

$$\frac{\partial h}{\partial b} = \frac{y}{\sqrt{1+Y^2}}\cdot f'(b). \tag{11}$$

Note that for $f(b) = b$, we have $f'(b) = 1$, and for $f(b) = e^b$, $f'(b) = f(b) = e^b$.

3 Likelihood for Incremental Normalization

Here, *incremental normalization* means that the model parameters $\mu_1, \dots, \mu_n$ and σ^2 are already known from a fit to a previous set of μarrays, i. e. a set of reference arrays. See Section 4 for the profile likelihood approach that is used if $\mu_1, \dots, \mu_n$ and σ^2 are not known and need to be estimated from the same data. Versions ≥ 2.0 of the *vsn* package implement both of these approaches; in versions $1.X$ only the profile likelihood approach was implemented, and it was described in the initial publication [1].

First, let us note that the likelihood 6 is simply a product of independent terms for different i. We can optimize the parameters (a_i, b_i) separately for each $i = 1, \dots, d$. From the likelihood 6 we get the i-th negative log-likelihood

$$-\log(L) = \sum_{i=1}^{d} -LL_i \tag{12}$$

$$-LL_i = \frac{n}{2}\log\left(2\pi\sigma^2\right) + \sum_{k=1}^{n}\left(\frac{(h(y_{ki})-\mu_k)^2}{2\sigma^2} + \log\frac{\sqrt{1+Y_{ki}^2}}{f(b_i)}\right) \tag{13}$$

$$= \frac{n}{2}\log\left(2\pi\sigma^2\right) - n\log f(b_i) + \sum_{k=1}^{n}\left(\frac{(h(y_{ki})-\mu_k)^2}{2\sigma^2} + \frac{1}{2}\log\left(1+Y_{ki}^2\right)\right) \tag{14}$$

This is what we want to optimize as a function of a_i and b_i. The optimizer benefits from the derivatives. The derivative with respect to a_i is

$$\begin{aligned}\frac{\partial}{\partial a_i}(-LL_i) &= \sum_{k=1}^{n}\left(\frac{h(y_{ki})-\mu_k}{\sigma^2}+\frac{Y_{ki}}{\sqrt{1+Y_{ki}^2}}\right)\cdot\frac{1}{\sqrt{1+Y_{ki}^2}}\\ &= \sum_{k=1}^{n}\left(\frac{r_{ki}}{\sigma^2}+A_{ki}Y_{ki}\right)A_{ki}\end{aligned} \tag{15}$$

and with respect to b_i

$$\begin{aligned}\frac{\partial}{\partial b_i}(-LL_i) &= -n\frac{f'(b_i)}{f(b_i)}+\sum_{k=1}^{n}\left(\frac{h(y_{ki})-\mu_k}{\sigma^2}+\frac{Y_{ki}}{\sqrt{1+Y_{ki}^2}}\right)\cdot\frac{y_{ki}}{\sqrt{1+Y_{ki}^2}}\cdot f'(b_i)\\ &= -n\frac{f'(b_i)}{f(b_i)}+f'(b_i)\sum_{k=1}^{n}\left(\frac{r_{ki}}{\sigma^2}+A_{ki}Y_{ki}\right)A_{ki}y_{ki}\end{aligned} \tag{16}$$

Here, I have introduced the following shorthand notation for the "intermediate results" terms

$$r_{ki} = h(y_{ki})-\mu_k \tag{17}$$

$$A_{ki} = \frac{1}{\sqrt{1+Y_{ki}^2}}. \tag{18}$$

Variables for these intermediate values are also used in the C code to organise the computations of the gradient.

4 Profile Likelihood

If $\mu_1,\ldots,\mu_n$ and σ^2 are not already known, we can plug in their maximum likelihood estimates, obtained from optimizing LL for $\mu_1,\ldots,\mu_n$ and σ^2:

$$\hat\mu_k = \frac{1}{d}\sum_{j=1}^{d}h(y_{kj}) \tag{19}$$

$$\hat\sigma^2 = \frac{1}{nd}\sum_{k=1}^{n}\sum_{j=1}^{d}(h(y_{kj})-\hat\mu_k)^2 \tag{20}$$

into the negative log-likelihood. The result is called the negative profile log-likelihood

$$-PLL = \frac{nd}{2}\log\left(2\pi\hat\sigma^2\right)+\frac{nd}{2}-n\sum_{j=1}^{d}\log f(b_j)+\frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{d}\log\sqrt{1+Y_{kj}^2}. \tag{21}$$

Note that this no longer decomposes into a sum of terms for each j that are independent of each other – the terms for different j are coupled through Equations 19 and 20. We need the following derivatives.

$$\begin{aligned}\frac{\partial\hat\sigma^2}{\partial a_i} &= \frac{2}{nd}\sum_{k=1}^{n}r_{ki}\frac{\partial h(y_{ki})}{\partial a_i}\\ &= \frac{2}{nd}\sum_{k=1}^{n}r_{ki}A_{ki}\end{aligned} \tag{22}$$

$$\frac{\partial\hat\sigma^2}{\partial b_i} = \frac{2}{nd}\cdot f'(b_i)\sum_{k=1}^{n}r_{ki}A_{ki}y_{ki} \tag{23}$$

So, finally

$$\frac{\partial}{\partial a_i}(-PLL) = \frac{nd}{2\hat{\sigma}^2} \cdot \frac{\partial \hat{\sigma}^2}{\partial a_i} + \sum_{k=1}^{n} A_{ki}^2 Y_{ki}$$

$$= \sum_{k=1}^{n} \left(\frac{r_{ki}}{\hat{\sigma}^2} + A_{ki} Y_{ki} \right) A_{ki} \quad (24)$$

$$\frac{\partial}{\partial b_i}(-PLL) = -n \frac{f'(b_i)}{f(b_i)} + f'(b_i) \sum_{k=1}^{n} \left(\frac{r_{ki}}{\hat{\sigma}^2} + A_{ki} Y_{ki} \right) A_{ki} y_{ki} \quad (25)$$

5 Summary

Likelihoods, from Equations 12 and 21:

$$-LL_i = \underbrace{\frac{n}{2}\log\left(2\pi\sigma^2\right)}_{\text{scale}} + \underbrace{\sum_{k=1}^{n}\frac{(h(y_{ki})-\mu_k)^2}{2\sigma^2}}_{\text{residuals}} \underbrace{-n\log f(b_i) + \frac{1}{2}\sum_{k=1}^{n}\log(1+Y_{ki}^2)}_{\text{jacobian}} \tag{26}$$

$$-PLL = \underbrace{\frac{nd}{2}\log\left(2\pi\hat{\sigma}^2\right)}_{\text{scale}} + \underbrace{\frac{nd}{2}}_{\text{residuals}} + \underbrace{\sum_{i=1}^{d}\left(-n\log f(b_i) + \frac{1}{2}\sum_{k=1}^{n}\log(1+Y_{ki}^2)\right)}_{\text{jacobian}} \tag{27}$$

The computations in the C code are organised into steps for computing the terms "scale", "residuals" and "jacobian".

Partial derivatives with respect to a_i, from Equations 15 and 24:

$$\frac{\partial}{\partial a_i}(-LL_i) = \sum_{k=1}^{n}\left(\frac{r_{ki}}{\sigma^2} + A_{ki}Y_{ki}\right)A_{ki} \tag{28}$$

$$\frac{\partial}{\partial a_i}(-PLL) = \sum_{k=1}^{n}\left(\frac{r_{ki}}{\hat{\sigma}^2} + A_{ki}Y_{ki}\right)A_{ki} \tag{29}$$

Partial derivatives with respect to b_i, from Equations 16 and 25:

$$\frac{\partial}{\partial b_i}(-LL_i) = -n\frac{f'(b_i)}{f(b_i)} + f'(b_i)\sum_{k=1}^{n}\left(\frac{r_{ki}}{\sigma^2} + A_{ki}Y_{ki}\right)A_{ki}y_{ki} \tag{30}$$

$$\frac{\partial}{\partial b_i}(-PLL) = -n\frac{f'(b_i)}{f(b_i)} + f'(b_i)\sum_{k=1}^{n}\left(\frac{r_{ki}}{\hat{\sigma}^2} + A_{ki}Y_{ki}\right)A_{ki}y_{ki}. \tag{31}$$

Note that the terms have many similarities – this is used in the implementation in the C code.

References

[1] W. Huber, A. von Heydebreck, H. Sültmann, A. Poustka, and M. Vingron. Variance stablization applied to microarray data calibration and to quantification of differential expression. *Bioinformatics*, 18:S96–S104, 2002.

[2] W. Huber, A. von Heydebreck, H. Sültmann, A. Poustka, and M. Vingron. Parameter estimation for the calibration and variance stabilization of microarray data. *Statistical Applications in Genetics and Molecular Biology*, Vol. 2: No. 1, Article 3, 2003. http://www.bepress.com/sagmb/vol2/iss1/art3