

# Factors for Constructing Control Limits

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## Abstract

In this note, we provide several mathematical formulas of the factors which are used for constructing the control limits. These factors can be easily obtained by using the `factors.cc` function in the robust quality control chart (`rQCC`) R package.

## 1 Factors for computing control chart lines

In this section, we provide a brief summary of mathematical relations of factors for computing the control chart *lines*. For more details, see Supplement A of ASTM (STP 15-D) [1] and Supplement B of ASTM (STP 15-C) [2].

The mathematical relations for factors ( $c_2$ ,  $c_4$ ,  $d_2$ ,  $d_3$ ) are based on sampling randomly from a normal distribution. These are given by

$$\begin{aligned}c_2(n) &= \sqrt{\frac{2}{n}} \cdot \frac{\Gamma(n/2)}{\Gamma(n/2 - 1/2)}, \\c_4(n) &= \sqrt{\frac{2}{n-1}} \cdot \frac{\Gamma(n/2)}{\Gamma(n/2 - 1/2)}, \\d_2(n) &= 2 \int_0^\infty \left\{ 1 - [\Phi(z)]^n - [1 - \Phi(z)]^n \right\} dz,\end{aligned}$$

and

$$d_3(n) = \sqrt{E(R^2) - d_2(n)^2},$$

where  $\Phi(\cdot)$  is the cumulative distribution function (cdf) of the standard normal distribution and  $E(R^k)$  is given in (2) of Appendix B. All the detailed derivations of  $c_4(n)$  are provided in Appendix A and those of  $d_2(n)$  and  $d_3(n)$  are given in Appendix B. Note that  $c_2(n)$  has been used in ASTM (STP 15-C) [2] and it is replaced by  $c_4(n)$  in ASTM (STP 15-D) [1]. Thus,  $c_2(n)$  is rarely used after the year of 1976.

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## 2 Factors for computing control limits

The factors below are used for constructing a variety of control charts with the choice of  $g \cdot \sigma$  limits. For more details, see Supplement A of ASTM (STP 15-D) [1] and Supplement B of ASTM (STP 15-C) [2]. Note that the American Standard uses  $3 \cdot \sigma$  limits with 0.27% false alarm rate, while the British Standard uses  $3.09 \cdot \sigma$  limits with 0.20% false alarm rate.

- For averages:

$$\begin{aligned} A(n) &= \frac{g}{\sqrt{n}}, \\ A_1(n) &= \frac{g}{c_2(n)\sqrt{n}} = \frac{A(n)}{c_2(n)}, \\ A_2(n) &= \frac{g}{d_2(n)\sqrt{n}} = \frac{A(n)}{d_2(n)}, \\ A_3(n) &= \frac{g}{c_4(n)\sqrt{n}} = \frac{A(n)}{c_4(n)}. \end{aligned}$$

Note that  $A_1(n)$  in ASTM (STP 15-C) [2] was replaced by  $A_3(n)$  in ASTM (STP 15-D) [1] in the year of 1976. Since then,  $A_1(n)$  is rarely used.

- For standard deviations:

$$\begin{aligned} B_1(n) &= \max \left\{ c_2(n) - g \cdot \sqrt{\frac{n-1}{n} - c_2(n)^2}, 0 \right\}, \\ B_2(n) &= c_2(n) + g \cdot \sqrt{\frac{n-1}{n} - c_2(n)^2}, \\ B_3(n) &= \max \left\{ 1 - \frac{g}{c_4(n)} \cdot \sqrt{1 - c_4(n)^2}, 0 \right\}, \\ B_4(n) &= 1 + \frac{g}{c_4(n)} \cdot \sqrt{1 - c_4(n)^2}, \\ B_5(n) &= \max \left\{ c_4(n) - g \cdot \sqrt{1 - c_4(n)^2}, 0 \right\} = c_4(n) \cdot B_3(n), \\ B_6(n) &= c_4(n) + g \cdot \sqrt{1 - c_4(n)^2} = c_4(n) \cdot B_4(n). \end{aligned}$$

Note that  $B_1(n)$  and  $B_2(n)$  in ASTM (STP 15-C) [2] are replaced by  $B_5(n)$  and  $B_6(n)$ , respectively, in ASTM (STP 15-D) [1].

In ASTM (STP 15-C), however,  $B_3(n)$  and  $B_4(n)$  are defined as

$$\begin{aligned} B_3(n) &= \max \left\{ 1 - \frac{g}{c_2(n)} \cdot \sqrt{\frac{n-1}{n} - c_2(n)^2}, 0 \right\}, \\ B_4(n) &= 1 + \frac{g}{c_2(n)} \cdot \sqrt{\frac{n-1}{n} - c_2(n)^2}, \end{aligned}$$

which are easily obtained by  $B_1(n)/c_2(n)$  and  $B_2(n)/c_2(n)$  in ASTM (STP 15-C), respectively. Thus, we calculate  $B_3(n)$  and  $B_4(n)$  based only on ASTM (STP 15-D) [1] instead of ASTM (STP 15-C).

- For ranges:

$$\begin{aligned}
 D_1(n) &= \max \{d_2(n) - g \cdot d_3(n), 0\}, \\
 D_2(n) &= d_2(n) + g \cdot d_3(n), \\
 D_3(n) &= \max \left\{ 1 - g \cdot \frac{d_3(n)}{d_2(n)}, 0 \right\} = \frac{D_1(n)}{d_2(n)}, \\
 D_4(n) &= 1 + g \cdot \frac{d_3(n)}{d_2(n)} = \frac{D_2(n)}{d_2(n)}.
 \end{aligned}$$

- For individuals:

$$\begin{aligned}
 E_1(n) &= \frac{g}{c_2(n)}, \\
 E_2(n) &= \frac{g}{d_2(n)}, \\
 E_3(n) &= \frac{g}{c_4(n)}.
 \end{aligned}$$

Note that  $E_1(n)$  in ASTM (STP 15-C) [2] is replaced by  $E_3(n)$  in ASTM (STP 15-D) [1].

# Appendices

## A The bias correction factor for the sample standard deviation

It is well known that

$$E(S^2) = \sigma^2,$$

where  $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2 / (n-1)$ ,  $X_i \sim N(\mu, \sigma^2)$ , and  $\bar{X} = \sum_{i=1}^n X_i / n$ . It deserves mentioning that  $E(S) \neq \sigma$ .

Using the fact that  $Y = (n-1)S^2/\sigma^2$  has the chi-square distribution with  $n-1$  degrees of freedom which is equivalent to the gamma distribution with  $\alpha = (n-1)/2$  (shape) and  $\theta = 2$  (scale), we obtain the unbiased estimator of  $\sigma$ . Now, it is well known that

$$E[Y^c] = \frac{\Gamma(\alpha + c)\theta^c}{\Gamma(\alpha)},$$

when  $Y$  has the gamma distribution with shape  $\alpha$  and scale  $\theta$ . Clearly, for  $c = 1/2$ , we have

$$E[\sqrt{Y}] = \frac{\Gamma(\alpha + 1/2)\sqrt{\theta}}{\Gamma(\alpha)}.$$

Then we obtain

$$E[\sqrt{(n-1)S^2/\sigma^2}] = \frac{\Gamma(n/2)\sqrt{2}}{\Gamma(n/2 - 1/2)}.$$

This implies that

$$E(S) = c_4(n)\sigma,$$

where

$$c_4(n) = \sqrt{\frac{2}{n-1}} \cdot \frac{\Gamma(n/2)}{\Gamma(n/2 - 1/2)}.$$

Thus, the estimator  $S/c_4(n)$  is unbiased for  $\sigma$ .

## B The bias correction factors for the range

We can also estimate  $\sigma$  using the range,  $R = X_{(n)} - X_{(1)}$ , where  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$  are the order statistics of a random sample of size  $n$  from  $N(\mu, \sigma)$ . It is known that  $R = X_{(n)} - X_{(1)}$  by itself is *not* unbiased for  $\sigma$ . In this section, we provide the bias correction factor for the range to estimate  $\sigma$  so that  $E(R/d_2(n)) = \sigma$ . We also provide the bias correction factor defined by  $\text{Var}(R/d_3(n)) = \sigma^2$ . First, we provide the following theorems and lemmas which are needed to obtain  $d_2(n)$  and  $d_3(n)$ .

**Theorem 1.** *Let  $X_1, X_2, \dots, X_n$  be a random sample with continuous cdf  $F(x)$  and pdf  $f(x)$ . Let  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$  be the order statistics of a random sample. Then the joint pdf of  $U = X_{(i)}$  and  $V = X_{(j)}$  for  $1 \leq i < j \leq n$  is given by*

$$f_{(i,j)}(u, v) = \frac{n!}{(i-1)! \times 1! \times (j-i-1)! \times 1! \times (n-j)!} \times \\ \left[ F(u) \right]^{i-1} f(u) \left[ F(v) - F(u) \right]^{j-i-1} f(v) \left[ 1 - F(v) \right]^{n-j}$$

for  $-\infty < u < v < \infty$ .

*Proof.* For more details, refer to Theorem 5.4.6 in Casella and Berger [3].  $\square$

Let  $Z_1, Z_2, \dots, Z_n$  be a random sample from a standard normal distribution with the pdf  $\phi(z)$  and the cdf  $\Phi(z)$ . For notational convenience, we denote  $U = Z_{(1)}$  and  $V = Z_{(n)}$ . Using Theorem 1, we have the joint pdf of  $U$  and  $V$

$$f_{(1,n)}(u, v) = n(n-1) \phi(u) \phi(v) [\Phi(v) - \Phi(u)]^{n-2}.$$

The goal is to derive the distribution of the range of the sample,  $V - U = Z_{(n)} - Z_{(1)}$ . Next we consider the new random variables given by  $Y_1 = U$  and  $Y_2 = V - U$ . Notice that the random variable  $Y_2$  is the *range*. The inverse transforms are easily obtained by  $u = y_1$  and  $v = y_1 + y_2$ . Then the joint pdf of  $Y_1$  and  $Y_2$ , denoted by  $g(y_1, y_2)$ , is given by

$$g(y_1, y_2) = n(n-1) \phi(y_1) \phi(y_1 + y_2) [\Phi(y_1 + y_2) - \Phi(y_1)]^{n-2} |J|,$$

where  $-\infty < y_1 < \infty$ ,  $y_2 > 0$  and

$$J = \det \begin{pmatrix} \frac{\partial u}{\partial y_1} & \frac{\partial u}{\partial y_2} \\ \frac{\partial v}{\partial y_1} & \frac{\partial v}{\partial y_2} \end{pmatrix} = 1.$$

Thus, we have

$$\begin{aligned} g_2(y_2) &= \int_{-\infty}^{\infty} g(y_1, y_2) dy_1 \\ &= n(n-1) \int_{-\infty}^{\infty} \phi(y_1) \phi(y_1 + y_2) [\Phi(y_1 + y_2) - \Phi(y_1)]^{n-2} dy_1. \end{aligned} \quad (1)$$

Note that the cdf of  $Y_2$  can be easily obtained by

$$G_2(y_2) = n \int_{-\infty}^{\infty} \phi(y_1) [\Phi(y_1 + y_2) - \Phi(y_1)]^{n-1} dy_1.$$

Next we consider the  $k$ -th moment of the range which was provided by Harter [4]. We provide a detailed derivation here. Using the pdf of the range in (1), we can obtain the  $k$ -th moment of the range,  $Y_2 = Z_{(n)} - Z_{(1)}$ , by calculating the expectation as follows:

$$\begin{aligned} E(Y_2^k) &= \int_0^{\infty} y_2^k g_2(y_2) dy_2 \\ &= n(n-1) \int_0^{\infty} y_2^k \int_{-\infty}^{\infty} \phi(y_1) \phi(y_1 + y_2) [\Phi(y_1 + y_2) - \Phi(y_1)]^{n-2} dy_1 dy_2 \\ &= n(n-1) \int_{-\infty}^{\infty} \left\{ \int_0^{\infty} y_2^k [\Phi(y_1 + y_2) - \Phi(y_1)]^{n-2} \phi(y_1 + y_2) dy_2 \right\} \phi(y_1) dy_1. \end{aligned}$$

For notational convenience, we replace  $(y_1, y_2)$  with  $(x, r)$ . Then we have

$$E(R^k) = n(n-1) \int_{-\infty}^{\infty} \left\{ \int_0^{\infty} r^k [\Phi(x+r) - \Phi(x)]^{n-2} \phi(x+r) dr \right\} \phi(x) dx, \quad (2)$$

where  $R = Z_{(n)} - Z_{(1)}$ .

Clearly, the expression for the  $k$ -th moment of the range requires the evaluation of a complicated double integral. Fortunately, for the case where  $k = 1$  which is the expectation, we can derive an alternative formula involving only a *single* integral. The derivation of this formula will require the application of three different lemmas which we state and prove below.

**Lemma 2.** *Let  $X$  be a continuous random variable with cdf  $F(x)$ . If  $E(|X|^k)$  exists, then we have*

$$(i) \lim_{x \rightarrow \infty} x^k \{1 - F(x)\} = 0 \quad \text{and} \quad (ii) \lim_{x \rightarrow -\infty} |x|^k F(x) = 0.$$

*Proof.* (i) For  $x > 0$ , we have

$$0 \leq x^k \{1 - F(x)\} = x^k \int_x^{\infty} dF(t) = \int_x^{\infty} x^k dF(t) \leq \int_x^{\infty} t^k dF(t).$$

Now, if we can show that the last term  $\int_x^{\infty} t^k dF(t) \rightarrow 0$  in the limit as  $x \rightarrow \infty$ , then this will complete the proof because we just showed that  $0 \leq x^k \{1 - F(x)\} \leq \int_x^{\infty} t^k dF(t)$  for  $x > 0$ . To prove that  $\int_x^{\infty} t^k dF(t) \rightarrow 0$  in the limit as  $x \rightarrow \infty$ , we observe that

$$\int_x^{\infty} t^k dF(t) = \int_{-\infty}^{\infty} |t|^k dF(t) - \int_{-\infty}^x |t|^k dF(t) = E(|X|^k) - \int_{-\infty}^x |t|^k dF(t).$$

Since  $E(|X|^k)$  exists and  $\lim_{x \rightarrow \infty} \int_{-\infty}^x |t|^k dF(t) = E(|X|^k)$ , we have

$$\int_x^{\infty} t^k dF(t) = E(|X|^k) - \int_{-\infty}^x |t|^k dF(t) \rightarrow 0$$

in the limit as  $x \rightarrow \infty$ .

(ii) For  $x < 0$ , we have

$$0 \leq |x|^k F(x) = |x|^k \int_{-\infty}^x dF(t) = \int_{-\infty}^x |x|^k dF(t) \leq \int_{-\infty}^x |t|^k dF(t).$$

Now, if we can show that the last term  $\int_{-\infty}^x |t|^k dF(t) \rightarrow 0$  in the limit as  $x \rightarrow -\infty$ , then this will complete the proof because we just showed that  $0 \leq |x|^k F(x) \leq \int_{-\infty}^x |t|^k dF(t)$  for  $x < 0$ . To prove that  $\int_{-\infty}^x |t|^k dF(t) \rightarrow 0$  in the limit as  $x \rightarrow -\infty$ , we have

$$\int_{-\infty}^x |t|^k dF(t) = \int_{-\infty}^{\infty} |t|^k dF(t) - \int_x^{\infty} |t|^k dF(t) = E(|X|^k) - \int_x^{\infty} |t|^k dF(t).$$

Since  $E(|X|^k)$  exists and  $\lim_{x \rightarrow -\infty} \int_x^{\infty} |t|^k dF(t) = E(|X|^k)$ , we have

$$\int_{-\infty}^x |t|^k dF(t) = E(|X|^k) - \int_x^{\infty} |t|^k dF(t) \rightarrow 0$$

in the limit as  $x \rightarrow -\infty$ . □

**Lemma 3.** Let  $X$  be a continuous random variable with cdf  $F(x)$ . Then we have

$$E(X) = \int_0^{\infty} [1 - F(x) - F(-x)] dx.$$

*Proof.* We have

$$E(X) = \int_{-\infty}^0 x dF(x) + \int_0^{\infty} x dF(x) = \int_{-\infty}^0 x dF(x) - \int_0^{\infty} x d[1 - F(x)]. \quad (3)$$

Using the integration by parts, we have

$$\int_{-\infty}^0 x dF(x) = [xF(x)]_{-\infty}^0 - \int_{-\infty}^0 F(x) dx \quad (4)$$

and

$$\int_0^{\infty} x d[1 - F(x)] = [x\{1 - F(x)\}]_0^{\infty} - \int_0^{\infty} [1 - F(x)] dx. \quad (5)$$

Applying Lemma 2 to both (4) and (5), we have

$$\int_{-\infty}^0 x dF(x) = - \int_{-\infty}^0 F(x) dx \quad \text{and} \quad \int_0^{\infty} x d[1 - F(x)] = - \int_0^{\infty} [1 - F(x)] dx. \quad (6)$$

Substituting (6) into (3) gives

$$E(X) = \int_0^{\infty} [1 - F(x)] dx - \int_{-\infty}^0 F(x) dx.$$

Since  $\int_{-\infty}^0 F(x) dx = \int_0^{\infty} F(-x) dx$ , we have

$$E(X) = \int_0^{\infty} [1 - F(x) - F(-x)] dx.$$

which completes the proof.  $\square$

It should be noted that Lemma 3 is also valid for discrete random variables, but the proof is omitted.

**Lemma 4.** Let  $X_1, X_2, \dots, X_n$  be a random sample with the cdf  $F(x)$ . Let  $F_{(j)}(x)$  denote the cdf of the  $j$ -th order statistic  $X_{(j)}$ . Then we have

$$(i) F_{(n)}(x) = [F(x)]^n \quad \text{and} \quad (ii) F_{(1)}(x) = 1 - [1 - F(x)]^n. \quad (7)$$

*Proof.* The proof is omitted.  $\square$

**Theorem 5.** Let  $X_1, X_2, \dots, X_n$  be a random sample with cdf  $F(x)$ . Then the expectation of the range,  $R = X_{(n)} - X_{(1)}$ , is given by

$$E(X_{(n)} - X_{(1)}) = \int_{-\infty}^{\infty} \left\{ 1 - [F(x)]^n - [1 - F(x)]^n \right\} dx.$$

*Proof.* Using Lemma 3, we have

$$E(X_{(n)}) = \int_0^{\infty} [1 - F_{(n)}(x) - F_{(n)}(-x)] dx$$

and

$$E(X_{(1)}) = \int_0^{\infty} [1 - F_{(1)}(x) - F_{(1)}(-x)] dx.$$

Applying (7) in Lemma 4 to the integral above, we obtain

$$E(X_{(n)}) = \int_0^{\infty} \{1 - [F(x)]^n - [F(-x)]^n\} dx,$$

and

$$E(X_{(1)}) = \int_0^{\infty} \{[1 - F(x)]^n dx - 1 + [1 - F(-x)]^n\} dx.$$

Thus, we have

$$\begin{aligned} E(X_{(n)} - X_{(1)}) &= \int_0^{\infty} \{1 - [F(x)]^n - [F(-x)]^n - [1 - F(x)]^n + 1 - [1 - F(-x)]^n\} dx \\ &= \int_0^{\infty} \{1 - [F(x)]^n - [1 - F(x)]^n\} dx + \int_0^{\infty} \{1 - [F(-x)]^n - [1 - F(-x)]^n\} dx. \end{aligned}$$

Using the change of the integration variable technique for the last term in the above, we have

$$\int_0^{\infty} \{1 - [F(-x)]^n - [1 - F(-x)]^n\} dx = \int_{-\infty}^0 \{1 - [F(x)]^n - [1 - F(x)]^n\} dx.$$

It is immediate from this result that we have

$$\begin{aligned} E(X_{(n)} - X_{(1)}) &= \int_0^{\infty} \{1 - [F(x)]^n - [1 - F(x)]^n\} dx + \int_{-\infty}^0 \{1 - [F(x)]^n - [1 - F(x)]^n\} dx \\ &= \int_{-\infty}^{\infty} \{1 - [F(x)]^n - [1 - F(x)]^n\} dx, \end{aligned}$$

which completes the proof.  $\square$

It should be noted that the above lemmas and theorems are also valid for non-normal distributions. But, we use the results specifically in the case of the normal distribution. Now suppose that we have a random sample from a *standard* normal distribution,  $Z_1, Z_2, \dots, Z_n$  and that we want to calculate the expectation of the sample range. Then we have

$$E(Z_{(n)} - Z_{(1)}) = \int_{-\infty}^{\infty} \{1 - [\Phi(z)]^n - [1 - \Phi(z)]^n\} dz.$$

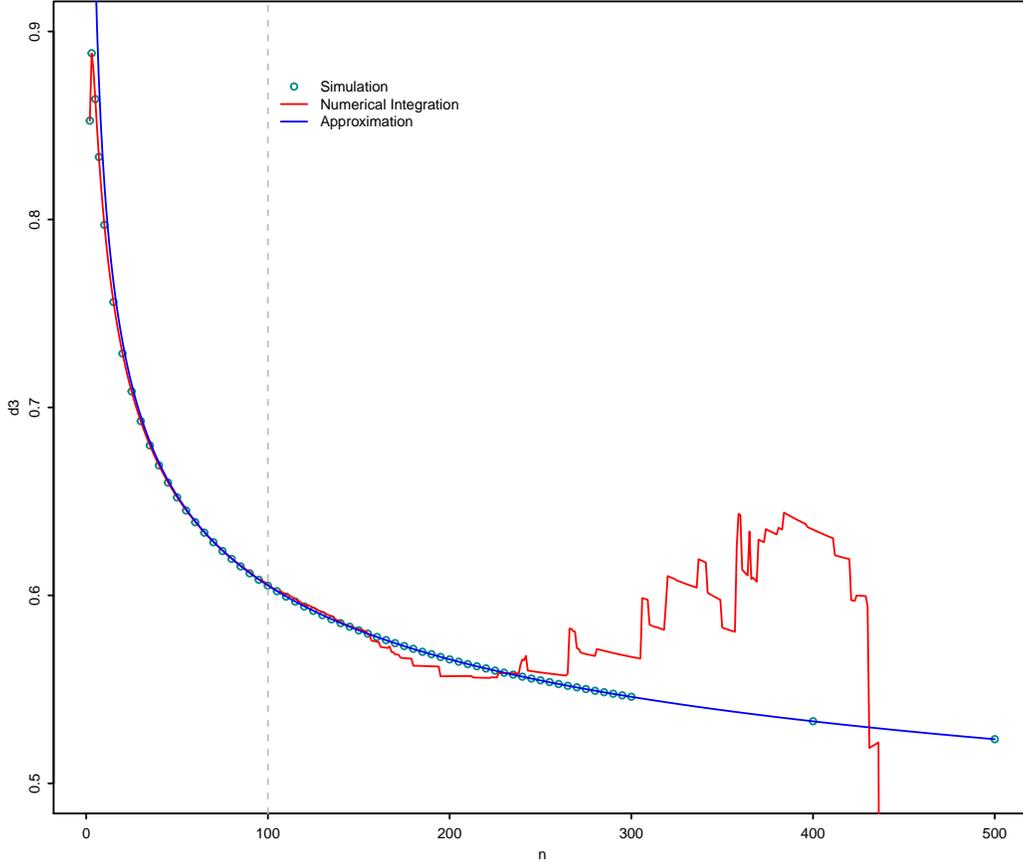


Figure 1: The plot of  $d_3(n)$  versus  $n$ .

Note that the integrand,  $1 - [\Phi(z)]^n - [1 - \Phi(z)]^n$ , is an even function due to the fact that  $\Phi(-z) = 1 - \Phi(z)$  which allows for the simplification of the expectation:

$$d_2(n) = E(Z_{(n)} - Z_{(1)}) = 2 \int_0^\infty \left\{ 1 - [\Phi(z)]^n - [1 - \Phi(z)]^n \right\} dz.$$

Thus, the estimator  $R/d_2(n) = (X_{(n)} - X_{(1)})/d_2(n)$  is unbiased for  $\sigma$  with  $X_i \sim N(\mu, \sigma^2)$ . Then it is easily seen that  $R = X_{(n)} - X_{(1)} = \sigma(Z_{(n)} - Z_{(1)})$ , where  $Z_i \sim N(0, 1)$ .

Next, we consider the factor  $d_3(n)$  which is defined by  $\text{Var}(R/d_3(n)) = \sigma^2$ . Then, using  $\text{Var}(R) = \sigma^2 \text{Var}(Z_{(n)} - Z_{(1)})$ , we have

$$d_3(n) = \sqrt{\text{Var}(R)} = \sqrt{E(R^2) - \{E(R)\}^2} = \sqrt{E(R^2) - d_2(n)^2},$$

where  $E(R^2)$  can be obtained by (2).

The value of  $d_3(n)$  is involved with the double integration as shown in (2). We calculated the value of  $d_3(n)$  using the numerical integration. This double integration is accurate for small

values of  $n$  (say,  $n < 100$ ). As seen in Figure 1, the numerical calculation of  $d_3(n)$  (denoted by red line) is not reliable especially for large values of  $n$ . We double-checked these values by comparing with those obtained by the Monte Carlo simulation. To obtain more reliable and accurate calculation of  $d_3(n)$  especially for large values of  $n$ , we calculated the value of  $d_3(n)$  by Monte Carlo simulation, we generated a sample of size  $n$  from  $N(0, 1)$  and calculated the range. We iterated this simulation one hundred million times ( $I = 10^8$ ) and then calculated the empirical variance of this range (denoted by a circle in Figure 1).

We also obtained the approximation of  $d_3(n)$  by using the least squares method with the empirical variances of the ranges, which is given by

$$d_3(n) \approx \exp\left(0.73784298 + 0.06390565 \cdot \ln(n) - 0.71491753 \cdot \sqrt{\ln(n)}\right). \quad (8)$$

Figure 1 clearly shows that this approximation is very close to the simulated value of  $d_3(n)$  for large values of  $n$ . It is worth mentioning that The `factors.cc` function calculates  $d_3(n)$  using the numerical integration for  $n \leq 100$  and using the approximation in (8) for  $n > 100$ .

## References

- [1] ASTM Committee E-11. *Manual on Presentation of Data and Control Chart Analysis (STP 15-D)*. American Society for Testing and Materials, Philadelphia, PA, 4th edition, 1976.
- [2] ASTM Committee E-11. *ASTM Manual on Quality Control of Materials (STP 15-C)*. American Society for Testing and Materials, Philadelphia, PA, 1951.
- [3] G. Casella and R. L. Berger. *Statistical Inference*. Duxbury, Pacific Grove, CA, second edition, 2002.
- [4] H. Leon Harter. Tables of range and Studentized range. *The Annals of Mathematical Statistics*, 31(4):1122–1147, 1960.